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ON CONTROL OF TIME FOR REACHING A DOMAIN BY RANDOM MOTION *

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The problem is examined of maximizing the time for reaching the boundaries of the domain of a dynamic system moving under constant random perturbations of Gaussian white noise type and under pulsed action simulated by a Poisson process. Certain properties of the maximum time and of the optimal control are studied. The controlled motion of a rigid body acted on by random perturbing moments relative to the center of mass is examined as an example. The presence of random perturbations causes the kinetic moment of the rigid body to take arbitrarily large values. The purpose of controlling the rigid body's motion is to maximize the time for the kinetic moment to achieve some limit admissible value. The problem of retaining the system within a specified domain of phase coordinates has been taken up in a number of papers. In /1/ it was shown that certain questions on the motion control of an artificial satellite on synchronous orbits reduce to the maximization of the probability of retaining the system within a specified domain during a prescribed time interval. Another example of similar kind is connected with the working of water storage basins mentioned in /2/.

1. Consider the controlled stochastic system

$$dx(t) = [b(x,(t)) + c(x(t))\,u(x(t))]\,dt + \sigma(x,(t))\,d\xi(t) + f(x(t))\,d\eta(t),\quad t \geqslant 0,\quad x(0) = x \tag{1.1}$$

Here $x(t)$ is a vector in the n-dimensional Euclidean space E_n; $\xi(t)$ is an m-dimensional Wiener process; vector $\eta(t) = (\eta^1, \ldots, \eta^r)$; $\eta^i(t)$ $(i = 1, \ldots, r)$ is a homogeneous Poisson process with parameter λ_i and all the η^i are independent and in aggregate do not depend on process $\xi(t)$; control $u(x)$ is a vector with values in a closed bounded convex set $U \subset E_l$; b, $c(x)$, $\sigma(x)$ and $f(x)$ are certain deterministic matrices. Equation (1.1) should be understood in the Itô sense /3/. The latter paper also gave sufficient existence and uniqueness conditions for solutions of similar equations. Such conditions are, for example, a local Lipschitz condition for matrices c, σ and f and for the components of vectors b and u, as well as that their growth at infinity is no more than linear. Under such assumptions on the smoothness of the coefficients Eq. (1) with any initial condition $x(0) = x \in E_n$ has with unit probability a unique right-continuous solution free of discontinuities of the second kind, defining a homogeneous Markov process X. The last summand in (1.1) then characterizes shock perturbations.

Let G be a bounded domain in E_n with a sufficiently smooth boundary Γ. The control $u(x) \in U$ ensuring the existence of a solution of Eq. (1.1) with any initial condition $x(0) \in G$ up to the instant of exist of this solution from G is said to be admissible. The class of admissible controls is denoted U_0. In order to emphasize the dependence of process (1.1) on control u we shall sometimes denote it X_u and the corresponding trajectory by $x_u(t)$. Let $\tau_u = \tau_u(G)$ be the first exist instant of process X_u from G /4/, $u \in U_0$. We set

$$V_u(x) = M_x\tau_u,\quad V_0(x) = \sup_{u\in U_0} V_u(x)$$

Here M_x is the symbol for the mean computed under the condition that the initial state of the process is an arbitrary point $x \in G$. The problem is to find a control u_0 under which the equality

$$V_{u_0}(x) = M_x\tau_{u_0} = V_0(x) \tag{1.2}$$

holds for any $x \in G$. Sufficient conditions for solving the optimal problem indicated can be stated in terms of the Bellman equation corresponding to system (1.1). By L_u we denote the generating operator /4/ of process X_u in space $C^2(E_n)$, equal to /3/

$$L_uV(x) = \frac{1}{2}\sum_{i,j=1}^{n} a^{ij}(x)\,V_{x^ix^j} + (c(x)\,u(x) + b(x))'\,V_x + \sum_{i=1}^{r}[V(x + f(x)\,e_i) - V(x)]\,\lambda_i \tag{1.3}$$

$$x = (x^1, \ldots, x^n),\quad V(x) \in C^2(E_n)$$

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Here the prime denotes transposition, V_x is a vector with components $\partial V / \partial x^i$, a^{ij} is an element of matrix $a(x) = \sigma(x)\sigma'(x)$, e_i is an r-dimensional vector whose i-th component equals one while the rest are zero, $V_{x^i x^j}$ is a matrix with elements $\partial^2 V / \partial x^i \partial x^j$. By $D(G)$ we denote the space of functions twice continuously differentiable in G and continuous and bounded everywhere in E_n.

Lemma 1. Assume that a control $u_0(x) \in U_0$ and a nonnegative function $V(x) \in D(G)$ satisfying the conditions

$$L_{u_0} V(x) + 1 = 0, \quad L_u V(x) + 1 \leqslant 0, \ \forall x \in G, \ u \in U \qquad (1.4)$$
$$V(x) = 0, \quad x \,\overline{\in}\, G$$

have been found. Then u_0 is the optimal control and V is the Bellman function.

The proof in case $V \in C^2(E_n)$ is carried out by applying Itô's formula /3/ to function V at the instant $\tau_u(t) = \min(\tau_u, t)$, with a subsequent passage to the limit as $t \to \infty$ and the use of relations (1.4), which is consistent with the proof of the analogous statements in /5,6/. Let us explain this in more detail. From Itô's formula and the first two of conditions (1.4) it follows that

$$M_x V(x(\tau_u(t))) - V(x) = M_x \int_0^{\tau_u(t)} L_u V(x(s)ds) \leqslant -M_x \tau_u(t)$$

for any admissible control u. Consequently,

$$M_x \tau_u(t) \leqslant -M_x V(x(\tau_u(t))) + V(x) \leqslant 2 \sup_{x \in G} V(x)$$

The sequence $\tau_u(t)$ is nonnegative and monotonically nondecreasing in t. Therefore, by Lebesgue's theorem it is possible to pass to the limit as $t \to \infty$ under the integral sign. Doing this, we get that

$$M_x \tau_u \leqslant 2 \sup_{x \in G} V(x)$$

for any admissible control u. From this estimate Itô's formula and the first of relations (1.4) we conclude that

$$M_x V(x(\tau_{u_0})) - V(x) = -V(x) = -M_x \tau_{u_0}$$

Similarly, using the second of relations (1.4), we have

$$-V(x) \leqslant -M_x(\tau_u)$$

for any admissible control u. By the same token we have established the optimality of control u_0 when $V(x) \in C^2(E_n)$. In the general case it is sufficient to approximate $V \in D(G)$ by a uniformly bounded sequence of functions $V_i \in C^2(E_n)$ coinciding with V in some subdomain $G_i \subseteq G$, $G_i \uparrow G$, and to apply Itô's formula to these functions at instant $\min(\tau_u(G_i), t)$.

The following conditions in terms of the coefficients of (1.1), under which the stated means are finite, follows from /7,8/.

Lemma 2. Assume that for some $i = 1, \ldots$ at least one of the conditions

1) $a^{ii}(x) \geqslant a > 0$,

2) $h^i(x) \geqslant h > 0$ (or $h^i(x) \leqslant -h < 0$), where a and h are certain constants and $h^i(x)$ is the i-th component of vector

$$h(x) = b(x) + c(x)u(x) + \sum_{k=1}^{r} f(x)\lambda_k e_k$$

is fulfilled for all x from some neighborhood of the closure $\overline{G}$ of set G. Then the inequality

$$P_x\{\tau_u > t\} \leqslant ce^{-\nu t}, \quad \forall x \in G, \quad t \geqslant 0 \qquad (1.5)$$

is valid. Here P_x is the probability of the event within the braces under the condition that the initial state $x(0) = x$, while the positive constants c and ν are determined by the parameters of the original control problem and are independent of both x and u.

We rewrite relations (1.4) as the boundary-value problem

$$\sup_{u \in U} L_u V(x) = -1, \quad x \in G \qquad (1.6)$$
$$V(x) = 0, \quad x \,\overline{\in}\, G$$

At first we derive the existence conditions for the solution of the equation obtained from (1.6) if we omit the supremum sign in the first relation. As usual /9,10/ $C^{(k+\alpha)}(\overline{G})$ denotes the space of functions k times differentiable in $\overline{G}$, whose k-th derivatives satisfy a Hölder condition with index α $(0 < \alpha \leqslant 1)$. Also, we take the boundary Γ to be of class $C^{k+\alpha}$ if in some neighborhood W of each of its points the boundary Γ can be defined by an equation of form $x^i = \psi(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n)$, where $\psi \in C^{k+\alpha}(W \cap G)$.

Lemma 3. Assume that the elements of matrices $a(x)$ and $f(x)$ satisfy a Lipschitz condition in $\overline{G}$, the components of vector $b_u(x) = b(x) + c(x)u(x)$ are measurable and bounded, and the boundary $\Gamma \in C^{2+\alpha}$. In addition, suppose that a constant $\mu > 0$ has been found such that

$$\sum_{i,j=1}^{n} a^{ij}(x) y_i y_j \geqslant \mu \sum_{i=1}^{n} y_i^2$$

for any real numbers $y_1, \ldots, y_n$ and for all x. Then a unique almost-everywhere solution of the problem $(0 < \delta < 1)$

$$L_u V(x) = -1, \quad x \in G \qquad (1.7)$$
$$V(x) = 0, \quad x \overline{\in} G$$

exists in the class B of functions $V(x) \in C^{1+\delta}(G)$ having a square-integrable second derivative in G. We remark that a number of properties of operators L and V have been established in /11/.

Proof. On the basis of /12/ the boundary-value problem (1.7) has a unique solution $V(x) \in W_{p,0}^2(G)$ with $p \geqslant n$, and

$$V(x) = M_x \tau_u, \quad x \in G \qquad (1.8)$$

Here $W_{p,0}^2(G)$ is the space of measurable functions having generalized derivatives in the sense of Sobolev of up to second order, inclusive, summable on G to degree p, and vanishing for $x \overline{\in} G$. Let D be the differential part of operator (1.3). By $\omega(x)$ we denote a solution of the boundary-value problem

$$D\omega(x) - \lambda\omega(x) = -1 - \sum_{j=1}^{r} \lambda_j V(x + f(x) e_i), \ x \in G \qquad (1.9)$$
$$\omega(x) = 0, \quad x \in G, \quad \lambda = \lambda_1 + \ldots + \lambda_r \geqslant 0$$

From (1.5) follows the uniform boundedness in $x \in G$ of the functions $M_x \tau_u$. Hence from (1.8) it follows that the right hand side of (1.9) is measurable and bounded in $x \in G$. Hence /9/, boundary-value problem (1.9) has the unique solution $\omega(x)$ in class B. This solution $\omega(x)$ belongs as well to $W_{p,0}^2(G)$. In addition, in class $W_{p,0}^2(G)$ the function $V(x)$ too serves as a solution of problem (1.9). The equality $\omega(x) = V(x)$ is valid because the solution of problem (1.9) is unique in class $W_{p,0}^2(G)$. Hence, the solution $V(x)$ of problem (1.7) belongs to class B. Lemma 3 has been proved.

We take an arbitrary measurable function $u_1 = u_1(x)$ with values in set U, under the assumption that the hypotheses of Lemma 3 are fulfilled. By $V_1(x)$ we denote a solution of the equation $L_{u_1} V_1(x) = -1$ for almost all $x \in G$ with boundary condition $V_1(x) = 0$, $x \overline{\in} G$. We find a function $u_2(x)$ from the relation

$$\sup_{u \in U} u'c'(x) V_{1x} = u_2'(x) c'(x) V_{1x}$$

It is well known /13/ that vector $u_2(x)$ can be defined in such a way that its components are bounded measurable functions. Proceeding by induction, we construct the sequences $u_i(x) \in U$ and $V_i(x) \in B$ satisfying the conditions

$$L_{u_i} V_i(x) = -1, \quad x \in G \qquad V_i(x) = 0, \quad x \overline{\in} G \qquad (1.10)$$

$$\sup_{u \in U} u'c'(x) V_{ix} = u_{i+1}'(x) c'(x) V_{ix}$$

for all $i \geqslant 1$. We remark that the successive approximation procedure presented was applied in a number of papers (see /6,13/, for instance) for other stochastic system control problems. We set $\omega = V_{i+1} - V_i$. From relations (1.10) follow

$$L_{u_{i+1}} \omega(x) \leqslant 0, \quad x \in G \qquad \omega(x) = 0, \quad x \overline{\in} G \qquad (1.11)$$

Hence by virtue of /12/

$$\omega(x) = -M_x \int_0^{\tau_{u_{i+1}}} L_{u_{i+1}} \omega(x_{u_{i+1}}(t))\, dt \geqslant 0$$

Then

$$V_{i+1}(x) \geqslant V_i(x) \geqslant 0 \qquad (1.12)$$

Lemma 4. Let the hypotheses of Lemma 3 be fulfilled. Then as $i \to \infty$ the sequence (V_i, V_{ix}) converges to the function (V_0, V_{0x}) uniformly in $x \in G$. Function $V_0(x) \in B$. The second derivatives of functions V_i converge weakly to the second derivatives of function $V_0(x)$ in the space B_1 of functions square-integrable in G.

Proof. From formula (1.8) with $u = u_i$ and (1.5) it follows that

$$\sup_{i,\, x \in \bar{G}} V_i(x) \leqslant c < \infty \qquad (1.13)$$

Now we form a sequence of functions $\omega_i(x)$, the solutions of boundary-value problem (1.9) with $V = V_i$. Because of the uniqueness in class B (see Lemma 3) of the solution of problem (1.9) we have $\omega_i(x) = V_i(x)$. Hence, on the basis of /9/, the estimate

$$\sup_{i,\, x \in G} |\partial V_i / \partial x| \leqslant c < \infty$$

follows from (1.13). Hence from (1.12) it follows that $V_i(x)$ converges uniformly to $V_0(x)$. Therefore (for example, see Sect. 7 in /13/), $\omega_i(x)$ converges, together with its first derivatives, uniformly to function $V_0(x) \in B$, while the second derivatives of $\omega_i(x)$ converge weakly in B_1 to the second derivatives of function $V_0(x)$. Hence the validity of Lemma 4 follows from the equality $\omega_i(x) = V_i(x)$.

We set $u_0 = u_0(x)$ if

$$\sup_{u\in U} u'c'(x)V_{0x} = u_0'(x)c'(x)V_{0x}$$

Let us show that $V_0(x)$ is almost everywhere a solution of the Bellman Eq. (1.6). Indeed, we write (1.3) as

$$L_u V(x) = LV(x) + u'(x)c'(x)V_x$$

Using (1.10), for any i we obtain

$$l(x) = LV_0(x) + 1 + \sup_{u\in U} u'(x)c'(x)V_{0x} \geqslant \quad (1.14)$$
$$L(V_0(x) - V_i(x)) + u_i'(x)c'(x)(V_{0x}(x) - V_{ix}(x))$$

Let $i \to \infty$. Then the first parentheses in the right hand side of (1.14) tends to zero weakly in B_1, while the second, uniformly. Hence it follows that $l(x) \geqslant 0$ for almost all x. Further, making use of the obvious relation

$$u_0'(x)c'(x)V_{ix}(x) \leqslant u_{i+1}'(x)c'(x)V_{ix}(x)$$

and making simple manipulations, we find that

$$l(x) \leqslant u_{i+1}'(x)c'(x)(V_{ix} - V_{(i+1)x}) + \quad (1.15)$$
$$L(V_0(x) - V_{i+1}(x)) + u_0'(x)c'(x)(V_{0x} - V_{ix})$$

As before, we conclude that the right hand side of (1.15) converges weakly to zero. By the same token we have established that $l(x)$ equals zero almost everywhere. Thus, boundary-value problem (1.6) always has a solution $V_0(x) \in B$. In reality, however, as follows from /13/, function V_{0x} is of class $C^{2+\alpha}(G)$ for some α and $L_{u_0}V_0(x) = -1$ is valid for all $x \in G$. The last assertion stems from the Hölder-continuity of the expression $u_0'(x)c'(x)V_{0x}$. Speaking precisely, the conclusion that $V_0(x)$ is of class $C^{2+\alpha}(G)$ can be obtained from /13/ in the following manner. By W we denote a solution of the boundary-value problem

$$\sup_u DW(x) = r_0(x), \quad x \in G \quad (1.16)$$
$$W(x) = 0, \quad x \overline{\in} G$$

Here the known function $r_0(x)$ is determined by the formula

$$r_0(x) = -1 - \sum_{i=1}^{r} \lambda_i [V_0(x + f(x)e_i - V_0(x)] \quad (1.17)$$

We now note that the solution $W(x)$ of this boundary-value problem, by virtue of /13/, exists uniquely and is of class $C^{2+\alpha}$. In addition, we have $W = V_0$ by virtue of (1.17), (1.6) and the uniqueness in B of the solution of problem (1.16). Hence $V_0(x) \in C^{2+\alpha}(\overline{G})$. By Lemma 1 the inequality

$$M_x\tau_u(G) \leqslant V_0(x)$$

holds for any admissible control $u(x)$.

In the general case we can assert that an admissible control exists under which the error in the determination of the optimal value of the performance functional does not exceed a specified $\varepsilon > 0$. As a matter of fact, let $\varepsilon > 0$ be given. From /14/ (see the proofs of Lemmas 2.1 and 2.2) it follows that we can always find a vector-valued function $u(x)$ with components satisfying a Lipschitz condition, such that for all x

$$u'(x)c'(x)V_{0x} \geqslant u_0'(x)c'(x)V_{0x} - \varepsilon \quad (1.18)$$

where $V_0(x)$ is the solution of problem (1.6). Using (1.18) we obtain

$$L_u V_0(x) \geqslant -1 - \varepsilon \quad (1.19)$$

Using Itô's formula applied to V_0, from (1.19) we find

$$M_x\tau_u \geqslant V_0(x)(1 + \varepsilon)^{-1} \quad (1.20)$$

Noting that function $V_0(x)$ is bounded, we establish, in view of (1.20), that the difference $|M_x\tau_u - V_0(x)|$ is of the order of ε. The uniqueness of the solution of the Bellman equation can be established by making use of the preceding arguments. Indeed, let $V_1(x)$ and $V_2(x)$ be two solutions of problem (1.6), where $V_1(x) < V_2(x)$ for some $x \in G$. From what has been said follows the existence of an admissible control $u = u(x)$ such that

$$V_2(x)(1 + \varepsilon)^{-1} < M_x\tau_u(G) < V_1(x) \quad (1.21)$$

Because $\varepsilon > 0$ is arbitrary the statement to be proved follows from inequality (1.21). Thus, we have proved the following:

Theorem. Assume that the hypotheses of Lemma 3 are fulfilled. Then the boundary-value problem (1.6) has the unique solution $V_0(x)$ in class $C^{2+\alpha}(\overline{G})$. If $u_0(x)$ defined by formula (1.6) is an admissible control, then the functions $u_0(x)$ and $V_0(x)$ represent the optimal solution of the control problem for system (1.1) with performance index (1.2). In the general case an admissible control always exists, under which the performance index's optimal value can be approached to an arbitrary degree of accuracy.

2. Example. We consider the controlled motion of a rigid body relative to the center of mass under random perturbations of white noise type. By x we denote the kinetic moment of the rigid body, by a_i the moment of inertia, and by x_i the projections of vector x onto the principal central axes of inertia of the body. In the projections onto the axes mentioned the equations of motion have the form /15/

$$x_1^{\cdot} = x_2x_3(a_2 - a_3)(a_2a_3)^{-1} + u_1 + \sigma\xi_1^{\cdot} \quad (123) \tag{2.1}$$

Here (123) signifies that the equations for x_2 and x_3 are obtained from (2.1) by a cyclic permutation of subscripts, the constant $\sigma > 0$, and the control is subject to the constraint

$$|u| = \sum_{i=1}^{3} u_i^2 \leqslant b^2, \quad b > 0 \tag{2.2}$$

The mechanical sense of constraint(2.2) is that the controlling moment's action on the rigid body is the same in all directions /16/. It is required, by choosing the control, to maximize the mean of the first instant at which $|x| = r,\ r > 0$, under the condition that the modulus of the initial value $x(0)$ of the kinetic moment is less than r, i.e., $|x(0)| < r$. In this case Eq. (1.6) has the form $V(x) = 0,\ |x| = r$

$$0 = \frac{\sigma^2}{2}\sum_{i=1}^{3}\frac{\partial^2 V}{\partial x_i^2} + V_{x_1}x_2x_3(a_2 - a_3)(a_2a_3)^{-1} + \tag{2.3}$$

$$V_{x_2}x_1x_3(a_3 - a_1)(a_1a_3)^{-1} + 1 + V_{x_3}(a_1 - a_2)(a_1a_2)^{-1}x_1x_2 + \max_{u,\,|u| \leqslant b} u'V_x$$

The function

$$V_0(x) = 2\sigma^{-2}\int_{|x|}^{r} z(t)\,dt\int_0^t z^{-1}(s)\,ds \tag{2.4}$$

$$z(t) = \exp\left(2\int_t^r \sigma^{-2}(-b + \sigma^{-2}s)\,ds\right) \tag{2.5}$$

is a solution of this boundary-value problem. We substitute (2.4) into (2.3) and we find the control $u_0(x)$ maximizing the right hand side of (2.3). When $x \neq 0$ we have

$$u_0(x) = -bx\,|x|^{-1} \tag{2.6}$$

When $x = 0$ function $V_0(x)$ has a derivative equal to zero and, therefore, the control is not defined at $x = 0$. However, let us show that

1) if $x(0) \neq 0$, then under control (2.6) the probability that system (2.1) attains the state $x = 0$ before it reaches the surface $|x| = r$ equals zero;

2) if $x(0) = 0$, then under any admissible control the system (2.1) leaves the point $x = 0$ in an arbitrarily small admissible time, and, consequently, in view of 1), the system with probability one reaches the surface $|x| = r$ before it can return once again to the origin.

From statements 1) and 2) it follows that the control's value at $x = 0$ plays no role at all in the questions being examined. On the basis of 1) we take a number $\varepsilon > 0$ and we compute the probability $\omega(x)$ that system (2.1) reaches the surface $|x| = \varepsilon$ earlier than $|x| = r$ under control (2.6) and initial condition $x(0) = x$. Having written for $\omega(x)$ the appropriate Dirichlet problem, we obtain

$$\omega(x) = \int_{|x|}^{r} z(s)\,ds\left[\int_{\varepsilon}^{r} z(s)\,ds\right]^{-1}, \quad 0 < |x| \leqslant r \tag{2.7}$$

where function $z(t)$ is specified by equality (2.5). From (2.5) and (2.7) we conclude that $\omega(x) \to 0$ as $\varepsilon \to 0$.

We now prove statement 2). Let $u(x)$ be any admissible control and $\tau_u(\varepsilon)$ be the instant at which system (2.1) first leaves the sphere $|x| \leqslant \varepsilon$ under control u and zero initial condition. On the basis of /8/ (also see Lemma 2 of the present paper) $M\tau_u(\varepsilon) < \infty$. Therefore,

$$V_0(x(\tau_u(\varepsilon))) - V_0(0) = M\int_0^{\tau_u(\varepsilon)} L_uV_0(x_u(t))\,dt \leqslant M\int_0^{\tau_u(\varepsilon)} \sup_u L_uV_0(x_u(t))\,dt = -M\tau_u(\varepsilon)$$

Consequently, in view of (2.4) and the definition of $\tau_u(\varepsilon)$

$$M\tau_u(\varepsilon) \leqslant 2\sigma^{-2}\int_0^{\varepsilon} z(t)\,dt\int_0^t z^{-1}(s)\,ds$$

The validity of statement 2) follows from (2.5) and the arbitrariness of ε. Thus, we have established that in the rigid body motion control problem (2.1) the optimal control is given by formula (2.6) and the corresponding time by formula (2.4).

In the present paper we have studied certain control problems for systems being acted on by Gaussian and Poisson perturbations. Various aspects of the control of a diffusion systems are dealt with in /17,18/.

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